Finite Wing Theory

Session delivered by:

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Session Objectives

-- At the end of this session the delegate would have understood

• The finite wing theory
• Lifting line theory
• Elliptic wing results
• General lift distribution
• Induced drag
• Nonplanar wings
• Basics of Panel methods
• The concept wing sweep
Session Topics

1. Finite Wing Theory
2. Downwash
3. Lifting Line Theory
4. Elliptic Wing Results
5. General Lift Distribution
6. Induced Drag and Trefftz Plane
7. Far Field Analysis
8. Nonplanar Wakes
9. Munk’s Stagger Theorem
10. Nonplanar Wings
11. Introduction to Panel Methods
12. Weissinger Method
13. Simple Sweep Theory
14. Forward Sweep and Oblique Wing
Finite Wing Theory

This section deals with several aspects of wing theory, from the development of theoretical models of the finite wing to simple computational methods:

1. Wing Models
2. Lifting Line Theory
3. Induced Drag and the Trefftz Plane
4. Computational Models
5. Simple Sweep Theory
Wing Models

One may apply the results of 3-D potential theory in several ways. We first consider the theory of finite wings.

We might start out by saying that each section of a finite wing behaves as described by our 2-D analysis. If this were true then we would still find that the lift curve slope was $2\pi$ per radian, that the drag was 0, and the distribution of lift would vary as the distribution of chord. Unfortunately, things do not work this way. There are several reasons for this:
One explanation is that the high pressure on the lower surface of the wing and the low pressure on the upper surface causes the air to leak around the tips, causing a reduction in the pressure difference in the tip regions. In fact, the lift must go to zero at the tips because of this effect. We will next see how and why we must model the 3-D wing differently from 2-D.
If we were to take the naive view that the 2-D model would work in 3-D, we might have the picture shown on the next slide. If each section had the distribution of vorticity along its chord that it had in 2-D, the lift would be proportional to the chord, and would not drop off at the tips as we know it must.

This sort of model does not conform to our physical picture of what happens at the wing tips. And indeed, it does not satisfy the equations of 3-D fluid flow. The reason that this does not work is that in this case the streamlines are not confined to a plane. They move in 3-D and the flow pattern is quite different.
We could go back to the governing equations and start simply with the Laplace equation. By superimposing known solutions we could obtain a simple model of a 3-D wing. We might start by superimposing vortices on the wing itself:

But this is no more than the strip theory model that did not work. The reason that this model (which seems just to be a superposition of known solutions) is not adequate is that it violates the governing equations in certain regions. The model does not satisfy the Helmholtz laws since vorticity ends in the flow near the tips.
Some additional requirements must be imposed on the model. The requirements for such a model are just the Helmholtz vortex theorems, discussed previously.

Our simple 3-D model above may be modified as shown below to satisfy the first of the Helmholtz theorems.
In fact, as can be seen from the picture here, this vortex model is not too far from reality.
The downwash field and the existence of trailing vortices are not just some strange mathematical result. They are necessary for the conservation of mass in a 3-D flow.

Air is pushed downward behind the wing, but this downward velocity does not persist far from the wing. Instead it must move outward. The outward-moving air is then squeezed upward outboard of the wing and the flow pattern shown above develops.
The trailing vortex is visualized by NASA engineers by flying an agricultural airplane through a sheet of smoke. The main effect of this vortex wake is to produce a downwash field on the wing.
Downwash

The downwash field has several very significant effects:

- It changes the effective angle of attack of the airfoil section. This changes the lift curve slope and has many implications.

- Induced drag: Lift acts normal to flow in 2D. This (induced drag) accounts for about 40% of the fuel used in a commercial airplane, and as much as 80% of the drag in the critical climb segments.

- Downwash produces interference effects that are important in the analysis of stability and control.
The magnitude of the downwash can be estimated using the Biot-Savart law, discussed previously.

When applied to our simple model with two discrete trailing vortices, the equation predicts infinite downwash at the wing tips, a result that is clearly wrong. In fact, the induced downwash is not even very large.

The failure of this simple model led Prandtl to develop a slightly more sophisticated one in 1918. Rather than representing the wing with just one horseshoe-shaped vortex, the wing is represented by several of them:
In this way the circulation on the wing can vary from the root to the tip. The strength of the trailing vortex filaments is related to the circulation on the wing then by:

$$\Gamma_{\text{wake}} = \Delta \Gamma_{\text{wing}}$$

A vortex is shed from the wing whenever the circulation changes.
In the limit as the number of horseshoe vortices goes to infinity, the trailing wake is a sheet of vorticity. The trailing vortex strength per unit length in the y direction (vorticity) is the derivative of the total circulation on the wing at that station. From this model, we can derive the basic relations for finite wings.

\[ \gamma(y) = \frac{dr}{dy} \]
The vorticity strength in the trailing vortex sheet is given by:

$$\gamma = \frac{d \Gamma}{dy}$$

and since the wing circulation changes most quickly near the tips, the trailing vorticity is strongest in this region. This is why we see tip vortices, and not a complete vortex sheet, as in this NASA photo of an F-111 in a 4-g turn. The vortices are visible in this picture because the low pressure in this region lowers the temperature and we see the condensed water vapour.
Tip vortices seen in a NASA photo of an F-111 in a 4-g turn
Lifting Line Theory

Basic Theory
We could try using 2-D flow results for each section, but correct them for the influence of the trailing vortex wake and its downwash. This is the idea of lifting line theory.

We use the 2-D result that:  \( C_l = 2 \pi \alpha \)

together with the relation:  \( 1 = \rho U_\infty \Gamma \)

to obtain:  \( \Gamma = \pi c U_\infty \alpha \)
But the angle of attack used here is reduced through the effects of downwash so that the effective angle of attack is the true angle* minus the downwash angle:

$$\alpha_{\text{effective}} = \alpha_{\text{geometric}} - \frac{W_{\text{ind}}}{V_{\infty}}$$

Where the induced downwash, $W_{\text{ind}}$, is given by the Biot-Savart Law:

$$W_{\text{ind}}(y) = -\frac{1}{4\pi U_{\infty}} \int_{-b/2}^{b/2} \frac{\left(\frac{dr}{dy}\right)_{\text{ind}}}{y-y'} dy'$$

* Note that what we have called the geometric angle of attack is just that for flat plates but it is, in general, the angle of attack from zero lift.
Combining the expression for gamma:

\[ \Gamma = \pi c U_\infty \alpha_{\text{effective}} \]

with the expression for the downwash angle:

\[ \alpha_{\text{effective}} = \alpha_{\text{geometric}} + \frac{1}{4\pi U_\infty} \int_{-b/2}^{b/2} \frac{(d\Gamma/dy)_\text{wing}}{y - y'} dy' = \frac{\Gamma}{\pi c U_\infty} \]

provides an integral equation for the circulation distribution along the wing.

Just as in thin airfoil theory, the integral equation can be solved by assuming a Fourier series representation for the distribution.
Substitution of the expression for circulation into the integral equation leads to:

\[ I(y) = \frac{4}{\pi b} \sum_{n=1}^{\infty} A_n \sin n\theta \quad \text{where} \quad y = \frac{b}{2} \cos \theta \quad \text{so} \quad \theta = 0 \text{ at the tip and } \pi/2 \text{ at the root} \]

\[
\Gamma = \frac{1}{\rho U_\infty} = \frac{4}{\pi b \rho U_\infty} \sum_{n=1}^{\infty} A_n \sin n\theta
\]

Substitution of the expression for circulation into the integral equation leads to:

\[
\alpha_{\text{geometric}} - \frac{1}{2 \pi U_\infty b} \int_0^\pi \frac{1}{\cos \theta - \cos \theta'} \frac{\partial I}{\partial \theta} \wedge n \sin \theta' \, d\theta' = \frac{2}{\pi^2 q b c} \sum_{n=1}^{\infty} A_n \sin n\theta
\]

\[
= \alpha_{\text{geometric}} + \frac{1}{q \pi^2 b^2} \int_0^\pi \sum_{n=1}^{\infty} n A_n \cos n\theta' \, \frac{\cos \theta - \cos \theta'}{\cos \theta - \cos \theta'} \, d\theta'
\]

where \( q = \frac{1}{2} (\rho V^2) \) = Dynamic pressure.
After integrating we have:

\[
\frac{2}{\pi^2 q b c} \sum_{n=1}^{\infty} A_n \sin n\theta = \alpha_{\text{geometric}} - \frac{1}{q \pi b^2} \sum_{n=1}^{\infty} n A_n \frac{\sin n\theta}{\sin \theta}
\]

The solution of this equation for all values of \( y \) is not quite so easy as in the case of thin airfoil theory where we could get closed form expressions for the \( A_n 's \). This is generally done numerically. However, several interesting and simple results appear from this analysis without ever actually computing the \( A_n 's \) from the distribution of local angle of attack. Some of these are discussed in the next section.
Elliptic Wing Results

If, for example, we represent the lift distribution with only a single term in the Fourier series, then:

\[ l(y) = \frac{4L}{\pi b} \sin \theta = \frac{4L}{\pi b} \sqrt{1 - \left(\frac{y}{b/2}\right)^2} \]

This represents an elliptic distribution of lift.

The downwash angle is, in this case:

\[ \epsilon = \frac{w_{\text{ind}}(y)}{v_\infty} = - \frac{1}{4\pi U_\infty} \int_{-b/2}^{b/2} \frac{(d\Gamma/dy)_{\text{wing}}}{y - y'} \, dy' \]

The integral is constant when \(|y| < b/2\).
In this domain:

\[ \varepsilon = \frac{L}{q \pi b^2} = \frac{C_L}{\pi AR} \]

NOTE: AR = \( b^2 / S \)

Since the downwash distribution is constant the \( C_1 \) distribution is just:

\[ C_1 = 2\pi \left( \alpha - \frac{C_L}{\pi AR} \right) \]

If the angle of attack is also constant along the wing (no twist) then the \( C_1 \) is constant and we have:

\[ C_L = \frac{1}{S} \int_{-b/2}^{b/2} C_1(y) c \, dy \]
Then in this case the section $C_1$ is equal to the wing $C_L$ and:

$$C_L = 2\pi (\alpha - \frac{C_L}{\pi AR})$$

OR:

$$C_L = \frac{2\pi AR}{AR + 2} \alpha$$

Recall that this holds for unswept elliptical wings.
General Lift Distribution

If we are given the lift distribution we can compute the $A_n$'s as we would with any Fourier expansion. And once we know the Fourier coefficients, we may compute the downwash distribution and the induced drag:

$$
\epsilon = \frac{\nu_{ind}(y)}{V_\infty} = -\frac{1}{4\pi U_\infty} \int_{-b/2}^{b/2} \frac{(d\Gamma/dy)_{\Gamma_{in},p}}{y - y'} dy'
$$

$$
\Gamma = \frac{4}{\pi b p U_\infty} \sum_{n=1}^{\infty} A_n \sin n\theta
$$
Substitution and evaluation of the definite integral leads to:

\[ \varepsilon(\theta) = \frac{1}{q \pi b^2} \sum_n n A_n \frac{\sin n\theta}{\sin \theta} \]

This integral is sometimes called the Glauert integral and is tabulated, for example, in Kuethe and Chow.

This formula gives the downwash in the plane of the wing for arbitrary load distributions. For the simple elliptical case, closed form solutions for the downwash and sidewash at the start of the wake sheet exist.
The simple relation for the velocity induced by an elliptic wing tailing vortex sheet is:

$$W = v - iw = iw_0 \left[ \frac{Z}{\sqrt{Z^2 - (b/2)^2}} - 1 \right]$$

Here, the variable $Z$ is the complex coordinate $y + iz$ and $w_0$ is the downwash at the wing root: $y = z = 0$.

This formula permits computation of induced velocities behind a wing as they effect downstream surfaces such as horizontal tails.
Note that the downwash is only constant in the plane of the wing and behind the wing. As we move outboard of the wing or out of the plane of the wake, the downwash varies considerably and there is a rather large upwash beyond the wing tips.

This downwash field produces several important effects. It changes the lift of surfaces in other surface wakes. This is important in the analysis of airplane stability and the effectiveness of horizontal tails. As can be seen from the downwash plot, the interference of a canard wake with a wing is extreme: the wing lift is reduced behind the canard and the part of the wing outboard of the canard has increased lift.

The downwash also produces induced drag as discussed in the next section.
Induced Drag and the Trefftz Plane

Fundamentals
The 2-D paradox that surfaces in inviscid flow produce no drag no longer applies in 3-D. The downwash created by the trailing wake changes the direction of the force generated by each section:
In three dimensions the force per unit length acting on a vortex filament is

\[ \vec{F} = \rho \vec{V} \times \vec{\Omega} \]

Here, the local velocity \( V \) includes the component from the freestream and a component from the induced downwash. This latter component produces a component of force in the direction of the freestream: the induced drag.

The induced drag is related to the lift by:

\[ d \approx c \ell \]

From the results of lifting line theory for lift and downwash in terms of the Fourier coefficients of the lift distribution:

\[ l(y) = \frac{4}{\pi b} \sum_{n=1}^{\infty} A_n \sin n\theta \]
\[ \ell(\theta) = \frac{1}{\pi b^2} \sum_{n} n A_n \frac{\sin n\theta}{\sin \theta} \]
So we have the induced drag:

\[ D_i = \int_{\text{span}} d(y) \, dy = \frac{1}{q \, \pi \, b^2} \sum_n n \, A_n^2 \]

The induced drag is often written as:

\[ D_i = \frac{L^2}{q \, \pi \, b^2 \, e} \quad \text{with} \quad e = \frac{L^2}{\sum_n n \, A_n^2} \]
The induced drag may be written in coefficient form as:

\[ C_D = \frac{C_L^2}{\pi AR e} \]

with the same definition of \( e \). Note that \( e \) simply depends on the shape of the lift distribution. It is called the span efficiency factor or Oswald's efficiency factor. Note also that the induced drag force depends principally on the lift per unit span, \( L/b \).

We can determine quickly, from the expression for induced drag above that drag is a minimum for a given lift and span when all of the Fourier coefficients except the \( A_1 \) term (which produces lift) are zero. This corresponds to the elliptic loading case mentioned previously. In this case the downwash is constant and \( e = 1 \).
Far Field Analysis and the Trefftz Plane

The analysis above works quite well for analyzing the drag of wings given the distribution of lift. It was invented by Prandtl and Betz around 1920. It does involve a bit of hand-waving, though, and requires some very approximate idealizations of the flow. For example: when we talk about the downwash induced by the wake on the wing, just where on the wing do we mean? We assume a single bound vortex line and compute velocities there, but a real wing does not have a single bound vortex and the velocity induced by the wake varies along the chord. Fortunately, the answers from this model are more general than the model itself appears. The induced drag formulas can also be derived from very fundamental momentum considerations.
If the box is made large, contributions from certain sides vanish. In the limit as the box sides go to infinity we obtain the following expressions for lift and drag:

\[
F = \frac{\rho}{2} \int \nabla^2 \hat{n} \, dS - \rho \int \nabla (\nabla \cdot \hat{n}) \, dS
\]

Integrating over the box:
In the expression for $D$, $\frac{1}{2} (\rho)$ term is missing.
Here, $u$, $v$, and $w$ are the perturbation velocities induced by the wing and its wake. Note that the drag only depends on the velocities induced in the "Trefftz Plane" -- a plane far behind the wing.

The drag can be expressed as the integral over the infinite plane of the perturbation velocities squared. But, using Gauss' theorem (derivation) it can be expressed as a line integral over the wake itself:

$$D_i = \frac{1}{2} \int_{\Gamma} \mathbf{v} \cdot d\mathbf{l}$$

This simplifies the calculation of the drag.
The normalwash, $V_n$, is just the downwash if the wake is flat, but the downwash far behind the wing, not at the wing itself.

$$D_i = \frac{P}{2} \int \Gamma W_{\text{wake}} \, dy = \frac{1}{2} \int \Gamma(y) \frac{W_{\text{wake}}}{U_\infty} \, dy$$

Thus we would obtain the same expression as from lifting line theory if the downwash due to the wake at the wing is half the downwash at infinity. This is indeed the case for unswept wings modeled with a lifting line.

Velocity induced here comes from semi-infinite vortex line.

Velocity induced here by vortex line extending To left and right.
Nonplanar Wakes

All of the comments above apply to nonplanar wings as well as to simple planar ones. But we must be careful about the assumed wake shape when evaluating forces in the far field. The integrals above actually give the force on the wing and wake combination. Of course, in reality, there is no force on the wake sheet, but if we assume a shape a priori, it is not likely to be a force-free wake. However, since forces act in a direction perpendicular to the vortex, extending wakes streamwise always yields a drag-free wake and nearly correct answers for drag using far field methods.

Far-field velocities can also be used to compute the lift. The results are subtle, but rather interesting.
Munk's Stagger Theorem
The result that the drag of a lifting system depends only on the distribution of circulation shed into the wake leads to some very useful results in classical aerodynamics.

Perhaps the most useful of these is called Munk's stagger theorem. It states that:

*The total induced drag of a system of lifting surfaces is not changed when the elements are moved in the streamwise direction.*

The theorem applies when the distribution of circulation on the surfaces is held constant by adjusting the surface incidences as the longitudinal position is varied.
Munk's Stagger Theorem (Contd)
This implies that the drag of an elliptically-loaded swept wing is the same as that of an unswept wing. It also is very useful in the study of canard airplanes for which the canard downwash on the wing is quite complicated. Moving the canard very far behind the wing does not change the drag, but makes its computation much easier. One may use the stagger theorem to prove several other useful results. One of these is the mutual induced drag theorem which states that: The interference drag caused by the downwash of one wing on another is equal to that produced by the second wing on the first, when the surfaces are unstaggered (at the same streamwise location).

These results are especially useful in analyzing multiple lifting surfaces.
Trefftz Plane Drag Derivation

Why does the contribution to drag from all but the front and back sides vanish? The pressure terms are clearly 0 since the faces are parallel to the x direction. But the momentum terms are not so easy. We argue that the contribution goes to zero as the walls recede faster than the area goes to infinity. Here is why: In the far field, the induced velocities of a lifting system may be represented as the velocities induced by a single transverse vortex filament and two trailing vortices. The two trailing vortices cancel each other out in the far field, leaving only the piece of bound vorticity. This piece induces a velocity that varies as $1/r^2$ while the area is increasing as $r^2$. But because the flux $(V \cdot n)$ through the top and bottom are opposite, no first order term exists on these surfaces.
Furthermore, this vorticity induces no $V \cdot n$ through the sides.

We start with the expression for the drag in terms of the perturbation velocities:

$$D = \iint_{\text{Trefftz Plane}} v^2 + w^2 \, dS$$

$\frac{1}{2} (\rho)$ term is missing.

This result comes directly from the application of conservation of momentum and the incompressible Bernoulli equation.

Actually, it also assumes that the wake extends infinitely far downstream and trails back from the wing in the freestream direction. If we did not go very far downstream or the wake were not assumed to be straight, the more general expression would be:
\[ D = \iint (v^2 + w^2 - u^2) \, dS \]

Planar \( \frac{1}{2} (\rho) \) term is missing.

But, if we assume that the streamwise perturbation velocities are small, great simplifications are possible:

\[ D = \frac{\rho}{2} \iint w^2 + v^2 \, dS = \frac{\rho}{2} \iint \frac{\partial \phi^2}{\partial z} + \frac{\partial \phi^2}{\partial y} \, dS \]

\[ = \frac{\rho}{2} \iint \nabla \phi \cdot \nabla \phi \, dS \]

Now:

\[ \nabla \phi \cdot \nabla \phi = \nabla \cdot (\phi \nabla \phi) - \phi \nabla^2 \phi \]

and outside the wake:

\[ \nabla^2 \phi = 0 \]
So:

\[ D = \frac{p}{2} \iint \nabla \cdot (\phi \nabla \phi) \, dS \]

Gauss' theorem states that:

\[ \int_S \nabla \cdot F \, dS = \oint_C F \cdot n \, dl \]

Then:

\[ D = \frac{p}{2} \oint_C (\phi \nabla \phi) \cdot n \, dl \]
The contour integral is taken as shown below:

We thus obtain:

\[ D = \frac{\rho}{2} \int_{\text{wake}} \Delta \phi \frac{\partial \phi}{\partial n} \, dl \]
The jump in potential at the location $y$ in the wake is just the integral of $V \cdot ds$ from a point above the wake to a point below. Since the normal velocity is continuous across the wake, the integral is just equal to the circulation enclosed in the loop. This is just the circulation on the wing at the point where this part of the wake left the trailing edge. Similarly, the derivative of $\varphi$ normal to the wake, is the induced normal wash, $V_n$.

So:

$$D_i = -\frac{\rho}{2} \int_{\Gamma} V_n \, dl$$

The last expression may be recognized as the result of lifting line theory, but it has been derived in a much more general way.
Trefftz Plane Lift Derivation

We have discussed the calculation of drag based on the velocities induced in the Trefftz plane, but can lift be calculated in a similar way?

The answer is not so easy. We start with the expression for force based on the momentum equation.
…Let's assume (naively, for now) that the contribution of each of these integrals goes to zero on each side of the box, except for the back side, as the dimensions of the box are increased. This leaves the contribution in the Trefftz plane due to the wake.

In the Trefftz plane, if we assume that all of the induced velocities are normal to the plane, the lift becomes:
The evaluation of this integral seems straightforward. But, it is not.
Consider the integral

\[ L = \rho \int \int U_{\infty} w \, dS = \rho U_{\infty} \int \int w \, dz \, dy \]

when the wake is modeled simply by a pair of vortices.

The induced velocity, \( w \), is given by:

\[ w = \frac{-\Gamma}{2\pi} \left\{ \frac{u - s}{(u-s)^2 + z^2} - \frac{u + s}{(u+s)^2 + z^2} \right\} \]

Thus, the inner part of the above integral becomes:
So, the integral for lift is:

\[
\int_{-\infty}^{\infty} w \, dz = \frac{-\Gamma}{2\pi} \left\{ \tan^{-1}\left( \frac{z}{y-s} \right) - \tan^{-1}\left( \frac{z}{y+s} \right) \right\}_{-\infty}^{\infty}
= \frac{-\Gamma}{2} \left( \text{sign}(y-s) - \text{sign}(y+s) \right) = \Gamma \text{ when } |y| < s \text{ and 0 when } |y| > s
\]

So, the integral for lift is:

\[
L = \int_{-s}^{s} \rho U_{\infty} \Gamma \, dy = -2s \rho U_{\infty} \Gamma = b U_{\infty} \rho \Gamma
\]

This looks right; however, let's consider the same integral when the order of integration is reversed:
\[ L = \rho \iint U_\infty \, w \, dS = \rho \int_\infty \int w \, dy \, dz \]

The inner part of the above integral now becomes:

\[
\int w \, dy = \frac{-\Gamma}{2\pi} \int_{-\infty}^{\infty} \left\{ \frac{Y-S}{(Y-S)^2 + Z^2} - \frac{Y+S}{(Y+S)^2 + Z^2} \right\} dy
\]

\[
= \frac{-\Gamma}{4\pi} \left\{ \ln \left[ \frac{(Y-S)^2 + Z^2}{(Y+S)^2 + Z^2} \right] \right\}_{-\infty}^{\infty}
\]

\[
= \frac{-\Gamma}{4\pi} \ln \left[ \frac{(Y-S)^2 + Z^2}{(Y+S)^2 + Z^2} \right]_{-\infty}^{\infty} = 0.
\]

This integrand is antisymmetric as shown in the plot below. So, the integral for lift is: \( L = 0. \)
Now we have a paradox. We get two values for the same integral. Actually, this is not a paradox; it is rather a function that is not Lebesgue integrable.
In order to evaluate an integral unambiguously, the function must satisfy two conditions:
1. It must be continuous, except at a countable number of points.
2. The integral of its absolute value must be finite.

To avoid this problem, the integral for lift can be first evaluated over finite limits. Taking \( z \) from -A to A and \( y \) from -B to B we find:

\[
L = \left( 2 \rho U \Gamma / \pi \right) \left\{ \left( B+s \right) \text{atan}\left( A/(B+s) \right) - \left( B-s \right) \text{atan}\left( A/(B-s) \right) \right\} \\
+ A/2 \ln\left[ \left( A^2+(B+s)^2 \right) / \left( A^2 + (B-s)^2 \right) \right]
\]
The limit as A and B get large depends on the ratio of A to B. When \( A >> B \) the value goes to \( 2 \rho U \Gamma s \), when \( B >> A \) the value is 0 and when \( A = B \) the value is \( \rho U \Gamma s \). Thus the integral remains ambiguous when evaluated over an infinite domain.

As noted by Larry Wigton of Boeing, this dilemma is resolved by using a different model of the flow field. When the vertical velocity associated with this vortex system is integrated over the Trefftz plane, no ambiguities arise. But, the results are surprising.
The result is that the contributions from the finite length trailing vortices goes to zero. The contribution from the bound vortex is found to be independent of the length of the trailing vortices and is:

\[ L_{\text{bound vortex contribution}} = \rho U_\infty R_s \]
The starting vortex contribution is similarly independent of the trailing vortex length and is equal to the bound vortex contribution. Thus, this lift is due to momentum flux, but not from the trailing vortices.

We finally need to look at how the pressure term on the upper and lower sides of the control volume is involved.

As might be expected, integrals once again are not unambiguous. They depend on the relative sizes of the box sides, even though everything is infinite.
A careful analysis leads to the following basic results:

1. If the wake length is small compared with the box width and height then the lift is associated with the momentum term of the starting and bound vortices.

2. If the wake length is large, and the box height is large compared with the width, then the lift is associated with the momentum term of the trailing vortices.

3. If the wake is long and the width is large compared with the height, then the lift is associated with the pressure terms on the top and bottom.
Nonplanar Wings

Even after the issues with infinite-domain integrals have been resolved, we must worry about the assumed wake position. Although we could argue that streamwise wakes can usually be used for far-field drag computations, streamwise wakes can still support lift forces. When the wing or wake is substantially nonplanar, these effects can be significant. In fact, the vortex lift generated by highly swept wings can be estimated by far-field methods only when the roll-up of the wake sheet is accurately computed. The alternatives in such cases are to use near field methods or to compute the wake shape.
Computational Models
Panel Methods

Many computational models and analysis methods are based on linear three-dimensional potential flow theory. These are discussed in the overview of panel methods in an earlier chapter.

In this section we take a look at the simplest panel method in more detail.
Weissinger Method

Weissinger theory or extended lifting line theory differs from lifting line theory in several respects. It is really a simple panel method (a vortex lattice method with only one chordwise panel), not a corrected strip theory method as is lifting line theory. This model works for wings with sweep and converges to the correct solution in both the high and low aspect ratio limits.

The version of this model used in the Wing Design program is actually a variant of Weissinger's method: it uses discrete skewed horseshoe vortices as shown.
Each horseshoe vortex consists of a bound vortex leg and two trailing vortices. This arrangement automatically satisfies the Helmholtz requirement that no vortex line ends in the flow. (The trailing vortices extend to infinity behind the wing.)
The basic concept is to compute the strengths of each of the "bound" vortices required to keep the flow tangent to the wing surface at a set of control points.

If the vortex of unit strength at station j produces a downwash velocity of $AIC_{ij}$ at station i, then the linear system of equations representing the boundary conditions may be written:
If the vortex of unit strength at station \( j \) produces a downwash velocity of \( AIC_{ij} \) at station \( i \), then the linear system of equations representing the boundary conditions may be written:

\[
[AIC] \{ \Gamma_i \} = U_\infty \{ \alpha \}
\]

where AIC is the influence coefficient matrix and \( \alpha \) represents the angle of incidence of the sections along the span (assumed a flat plate). It is a vector \( \alpha_i \). If the section has camber, the local angle of attack is taken as the angle from the zero lift line of the section.

The linear system of equations to be solved may also be written in terms of the angle of attack at the wing root and the twist amplitude.
For wings with a linear distribution of twist (washout):

\[
\{\alpha\} = \{\alpha_r\} - \frac{2\Theta}{b} \{y\}
\]

where:
\{\alpha_r\} is a vector containing the root angle of attack at each element

\{y\} is the spanwise coordinate, varying from 0 at the root to \(b/2\).

\{\Theta\} is the total twist (washout) in the wing from root to tip

Thus, the wing circulation distribution can be written as the sum of two distributions:

\[
\{\Gamma\} = [AIC]^{-1} \{\alpha_r\} \{1\} - \frac{2\Theta}{b} [AIC]^{-1} \{y\}
\]
Since the section lift (lift per unit length along the span) is related to the circulation by:

\[
\{l\} = \rho U_\infty \{\Gamma\}
\]

The lift distribution can be expressed as:

\[
\{l\} = \alpha_r \{l_1\} + \theta \{l_2\}
\]

where \(l_1\) and \(l_2\) are independent of the incidence angles and depend only on the planform shape of the wing.

Since the lift coefficient of the wing, \(C_L\), is linearly related to the angle of attack we can also write the lift distribution in the following form:

\[
\{l\} = C_L \{l_3\} + \theta \{l_4\}
\]
The first term is known as the additional lift distribution and the second term is called the basic lift distribution. They scale linearly with the wing lift coefficient and the twist angle respectively. Additional information on basic and additional lift distributions is available in the section on wing design.

An interactive computation based on this idea is available on the internet. Use it to investigate the effect of wing shape on lift distributions or to design wings as discussed in the following sections. The source code is available as well.
Wing Analysis Program

This Java application computes the lift and $C_l$ distribution over a wing with sweep and twist. To increase the angle of attack, click near the upper part of the plot; to reduce alpha, click in the lower area.

Details:
The analysis is a discrete vortex Weissinger computation. Pitching moment is based on the mean geometric chord and is measured about the root quarter chord point. The twist is assumed linear and is taken to be positive for washout (tip incidence less than root incidence).
Simple Sweep Theory
Lifting line theory works only for unswept wings.

Weissinger theory provides a means for computing the distribution of lift on swept wings, but not the chordwise distribution of pressures.

Vortex lattice models, panel methods, and nonlinear CFD provide pressure distributions on swept-back wings, but do not provide some of the insight that we can obtain with the simpler models.

It is mostly for this reason, and partly for historical reasons that simple sweep theory is interesting. It was invented by Busemann around 1935 and independently by R.T. Jones.
Consider an infinite wing as shown below.

If we ignore the effects of viscosity, and the wing is painted white so there is nothing to distinguish one section from another, we can slide the wing sideways and we could not tell that it was moving sideways. The air could not tell either, so the pressure distribution would remain unchanged.
We have just created an infinite, obliquely-swept wing that is moving with respect to the air at a speed:

\[ U_\infty^2 = U_1^2 + U_2^2 \]

We can use this fact to design a wing which can fly at a high speed with a pressure distribution associated with a lower speed.
The main idea behind sweeping the wing is to reduce the effects of compressibility. The component of the flow parallel to the wing is not affected by the presence of the wing; the normal component is decoupled from the tangential component.

This is true not only according to linear flow theory, but also in the case of nonlinear compressible flow with shock waves.
It is an interesting exercise to show how the full potential equations decompose into a normal term and a tangential term when one asserts that nothing changes in the tangential direction. This idea is called simple sweep theory. We can consider sections normal to the wing edges as operating in a flow with lower Mach number and dynamic pressure. The effective normal Mach number is then:

\[ M_\perp = M_\infty \cos \Lambda \]

but because of the reduced normal dynamic pressure, the section lift coefficient based on this component of the freestream velocity must be increased if the total lift is fixed:
Furthermore, at a given angle of attack, the lift is reduced. The reduction of lift at a given angle of attack for swept wings has important implications: the airplane incidence angle must be higher, causing several problems for some aircraft on landing approach (e.g. Concorde's drooped nose and long nose gear, F-8 variable incidence wing). The reduced lift curve slope due to sweep can improve the ride quality in gusty air, however.

This basic idea permits subsonic sections to be used at supersonic freestream Mach numbers or transonic airfoils to be used at Mach numbers higher than they would otherwise be able to operate. This works quite well even up to Mach numbers well over 1.0.
A recent airplane design had a design Mach number of 1.4. The airfoils were designed to operate at a Mach number of 0.7 (normal Mach number) with the wing swept 60°. Although the airplane lift coefficient was not meant to exceed 0.25, the airfoils had a design $C_l$ of 1.0 because of the reduced normal dynamic pressure from simple sweep theory.

The reason for designing a supersonic wing with sweep and subsonic airfoil sections can be seen in the results of thin airfoil theory which predicted no drag for subsonic sections, but did indicate that supersonic airfoil sections would produce drag due to thickness, camber, and lift.
Since the effect varies with cosine of the sweep angle, we expect that either forward, or aft swept wings would realize similar benefits. This is basically true, although, as discussed in the section on wing design and forward-swept wings, there are some important differences. Similarly, wings with oblique sweep have been designed and tested. Further discussions of oblique wings are given in the section on supersonic wings.
Forward-Swept Wings
Since sweep produces effects that vary with \( \cos(\text{sweep}) \), we might expect that either forward or aft sweep would yield the same results. To a first approximation, this is true; but, many other considerations can be important in comparing designs with forward and aft sweep. Historically these have led designers to adopt aft-swept wings for most aircraft, but this was not universally true. The Hansa Jet was a forward-swept wing business jet designed in the 1960's. Its forward swept wing permitted a larger cabin without a wing spar interrupting the floor. Some sailplanes have slight forward sweep to provide better visibility. Recently there has been renewed interest in the forward swept wing concept for aerodynamic reasons and a demonstrator / research aircraft, the X-29 was built by Grumman for NASA, DARPA (Defense Advanced Research Project Agency), and the Air Force.
The X-29 Aircraft with Forward-swept wings
Several aerodynamic advantages of the forward swept wing have been suggested. One of the more interesting of these is illustrated below. The claim is that the lower surface of a swept forward wing contributes a larger share of the total lift than the lower surface of an aft-swept wing.

Although exaggerated in this figure, this effect is predicted and observed. It is due in part to perturbation velocities induced by the 3-D thickness distribution and in part to the velocities induced by streamwise vorticity.
Advantages of forward sweep

- Better off-design span loading (but with less taper: $C_1$ advantage, weight penalty)
- Aeroelastically enhanced maneuverability
- Smaller basic lift distribution
- Reduced leading edge sweep for given structural sweep
- Increased trailing edge sweep for given structural sweep - lower $C_{Dc}$
- Unobstructed cabin
- Easy gear placement
- Good for turboprop placement
- Laminar flow advantages?
Disadvantages of forward sweep

- Aeroelastic divergence or penalty to avoid it
- Lower $|C_{1\beta}|$ (effective dihedral)
- Lower $C_{n\beta}$ (yaw stability)
- Bad for winglets
- Stall location (more difficult)
- Large $C_{m0}$ with flaps
- Reduced pitch stability due to additional lift and fuse interference
- Smaller tail length???
Oblique Wings

These results suggested by R.T. Jones that obliquely-swept wings would be the ideal shape for supersonic aircraft wings. He first proposed the concept in the 1940's and flew flying wing models at the first ICAS (Int. Council of the Aeronautical Sciences) meeting in Madrid in 1958. A great deal of work has been done since on oblique wing aircraft including design work by Boeing, General Dynamics, and Lockheed, wind tunnel testing and analysis by NASA, and flight testing of models and piloted aircraft.
The picture below shows the AD-1 low speed oblique wing demonstrator. Although one of the principal advantages involves reduced supersonic wave drag, the concept has other merits.
An all-wing version of the oblique wing was first proposed by G. H. Lee in 1962. The idea has been revived with the advent of active control systems and a recent **artist concept** of the oblique flying wing is shown below. Despite several questions about stability and control, this concept can be made to fly. (See the video clip)
Summary

The following topics were dealt in this session

• The finite wing
• Lifting line theory
• Elliptic wing results and general lift distribution
• Induced drag
• Nonplanar wings
• Basics of panel methods
• Forward sweep and oblique wing
Thank you