Three-Dimensional Potential Flow

Session delivered by:

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Session Objectives

-- At the end of this session the delegate would have understood

• Potential theory applied to 3-D irrotational flow
• Fundamental singularities in 3-D potential flow
• Helmholtz vortex theorems
• The Biot-Savart law
Session Topics

1. Three-dimensional Potential Flow
2. Fundamental Singularities in 3-D Potential Flow
3. Helmholtz Vortex Theorems
4. Biot-Savart Law
General Theory of 3-D Potential Flow
We start our discussion of three dimensional aerodynamics by considering the simple case of irrotational, inviscid flow. When compressibility can also be neglected, we consider solutions to Laplace's equation in 3D:

\[ \nabla^2 \phi = 0 \]

Just as in the 2-D case, since this equation is linear, we might construct solutions by superimposing known solutions. In 2-D we used sources and vortices extensively to construct the flow over airfoils. In 3-D we also use sources and vortices to model the flow over wings and bodies.

This section shows how this is done, starting with some results for some fundamental 3-D singularities.
Fundamental Singularities in 3D Potential Flow

One may derive fundamental solutions to Laplace's equation in 3-D, just as we did in 2-D (although complex variables are not quite so useful).

3-D Source

It was easily discovered that the potential: \( \phi = -k/r \) satisfied Laplace's equation in 3-D.

Since \( V = \text{grad} \, \phi \), the velocity associated with this solution is directed radially with a magnitude:
\[ V = \frac{k}{r^2}. \]

It is easily shown that the constant \( k \) is related to the volume flow rate, \( S \) by: \( k = S / 4\pi \), so:
\[ V = \frac{S}{4\pi} \frac{1}{r^2}. \]

The velocity distribution associated with this 3-D source dies off as \( r^2 \) rather than \( r \) as in the 2-D case.
**Point Doublet**

Another basic solution, that has been used with some success in supersonic aerodynamics programs is the point doublet, obtained by moving a point source and sink together while keeping the product of their strength, \( S \), and separation, \( L \), constant. With \( \mu = SL \), the velocity associated with the point doublet is:

\[
\vec{v} = \frac{\mu \cos \theta}{2 \pi r^3} \hat{r} + \frac{\mu \sin \theta}{4 \pi r^3} \hat{\theta}
\]
**Vortex Filament**

One of the most useful fundamental solutions to the 3-D Laplace equation is that of a vortex filament. A vortex filament may be visualized as a thin tube in which the flow has vorticity, $\omega$. In the limit as the diameter of the tube is made small, but the circulation, $\Gamma$, is held fixed, this region of vorticity is called a vortex filament.
Helmholtz Vortex Theorems

Helmholtz summarized some of the properties of vortex filaments, or vortices, in 1858 with his vortex theorems. These three theorems govern the behavior of inviscid three-dimensional vortices:

1. Vortex strength is constant
2. Vortices are forever (end on boundaries or form a closed path)
3. Vortices move with the flow

Vortex strength is constant: A vortex line in a fluid has constant circulation.

\[ \Gamma \]
This can be proved by imagining a closed 3-D loop around the vortex line as shown:

The integral around the closed loop from a to b to c to d to a cuts through no vorticity so from Stokes theorem the integral is zero. But as the slit is made very small the integral approaches the sum of the integral from b to c and the integral from d to a. These are the local circulations around the vortex line and so, the circulations must be constant along the line.
Since the vortex strength is constant along the vortex line the strength cannot suddenly go to zero. Thus, a vortex cannot end in the fluid. It can only end on a boundary or extend to infinity. Of course in an real, viscous fluid, the vorticity is diffused through the action of viscosity and the width of the vortex line can become large until it is hardly recognized as a vortex line. A tornado is an interesting example. One end of the twister is on a boundary; but at the other end, the vortex diffuses over a large area with vorticity.

As discussed in the section on sources and vortices, singularities such as vortices in the flow move along with the local flow velocity. Here, interactions of the vortices in the trailing wake, cause them to curve around each other and to form the nonplanar wake shown below.
Image from Head 1982 in van Dyke, An Album of Fluid Motion, used with permission.
**Biot-Savart Law**

The Biot-Savart law relates the velocity induced by a vortex filament to its strength and orientation. The expression, used frequently in electromagnetic theory, can be derived from the basic equations for the 3D potential. The result is:

\[
\mathbf{v} = \frac{1}{4\pi} \int \frac{\mathbf{r} \times \mathbf{F}}{|r|^3} \, ds
\]

In the simple case of an infinite vortex we obtain the 2-D result:

\[
|\mathbf{v}| = \frac{\Gamma}{2\pi r}
\]
In the case of a horseshoe vortex, the two trailing legs contribute:

\[
\psi(y) = \frac{\Gamma}{4\pi \left( \frac{b}{2} - y \right)} + \frac{\Gamma}{4\pi \left( \frac{b}{2} + y \right)}
\]

A simple subroutine is provided to compute the velocity components due to a vortex filament of length Gx, Gy, Gz with the start of the vortex rx, ry, rz from the point of interest.
Biot-Savart Law- Explanation

To determine the velocity associated with a vortex line, we consider the expression for vorticity:

\[ \omega = \nabla \times \mathbf{V} \]

If the flow is incompressible, then,

\[ \nabla \cdot \mathbf{V} = 0 \]

so we can write

\[ \mathbf{V} = \nabla \times \mathbf{A} \]

where \( \mathbf{A} \) is called the vector potential. We are free to choose \( \mathbf{A} \) so that it satisfies

\[ \nabla \cdot \mathbf{A} = 0 \]

\[ \omega = \nabla \times \mathbf{V} = \nabla^2 \mathbf{A} \]
Hence

$$\omega = \nabla \times V = -\nabla^2 A$$

This is a Poison equation for A which has the well-known solution:

$$A(r) = \frac{1}{4\pi} \int_{\text{spac}} \frac{\omega(s)}{|r-s|} \, dV$$

So the contribution of a length dl of vortex filament A is:

$$dA = \frac{1}{4\pi} \frac{\omega(s)}{|r-s|} (n \, ds \cdot dl)$$

Since, $\omega \cdot n \, ds = \Gamma$, then,

$$dA(r) = \frac{\Gamma}{4\pi} \frac{dl}{|r-s|}$$

and with a bit more algebra:

$$dV(r) = \frac{\Gamma}{4\pi} \frac{dl \times (r-s)}{|r-s|^3}$$
This expression may be integrated along the vortex line for the velocity induced by the filament to obtain the Biot-Savart law:

\[ d\vec{v}(r) = \frac{\vec{\Gamma}}{4\pi} \frac{d\vec{l} \times (r-s)}{|r-s|^3} \]

\[ \vec{v} = \frac{1}{4\pi} \int \frac{\vec{\Gamma} \times \vec{r}}{|r|^3} \, ds \]
Summary

The following topics were dealt in this session

- Potential theory applied to 3-D irrotational flow
- Fundamental singularities in 3-D potential flow: source, Doublet, Vortex filament
- Helmholtz vortex theorems
- The Biot-Savart law
Thank you